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INTRODUCTION TO TORIC MODIFICATIONS WITH AN APPLICATION TO REAL SINGULARITIES

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INTRODUCTION

One of motivations of this talk is to understand topological aspects of real polynomial-germs of three variables. If we draw a good picture of the zero-locus of a polynomial in a small ball in the real space, it assists working on the topology of them. The reader may expect there are no difficulties to draw a picture of the zero locus defined by relatively simple polynomials. But even if they defined by tetranomial like $z^5 + ty^6z + xy^7 + x^{15}$, the local pictures of zeros at the origin changes drastically when the parameter t passes through 0. S.Koike drew a picture of this polynomial in [12] with deep penetration on the location of arcs on its zero locus. This polynomial was known as the Briançon-Speder's family, which gives a μ-constant but not μ^*-constant family as complex polynomial-germs. See the page 12 in [4] for more information.

We prefer to draw pictures of zeros for such polynomials without much geometric intuition! We present a procedure to draw the picture of zero locus near the origin, and see that this is a routine for a relatively simple polynomials. If once we understand such a procedure, it becomes easy to understand what makes the picture looks different for e.g. Briançon-Speder family. The key is to understand a toric modification in a geometric way. Since it is easy to draw local pictures of nonsingular varieties, good geometric understanding of toric modification helps us to draw the pictures of singularities resolved by the toric modification.

We adopt here the definition of resolution into our situation from H.Hironaka's papers [8, page 142], and [9, page 459]. Let $f : U \to \mathbf{R}$ be a representative of a function-germ $(\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ on some neighborhood U of the origin of $\mathbf{R}^n$. We say that $\pi : X \to U$ gives a *resolution* of f if the following conditions are satisfied.

(i) X is non-singular.
(ii) π is proper and almost everywhere isomorphic.
(iii) $f \circ \pi$ defines locally everywhere normal crossings in X, that is, for each point ξ in X, there exists a local coordinate system $(y_1, ..., y_n)$ for X centered at ξ such that $f \circ \pi(y)$ is expressed by a monomial of y_i's.

In section 1, we present a brief review of toric modification with emphasizing geometric meaning. For complete treatment and more information about toric modification, the reader consults [3], [15] [10, 11], [6], and [1]. For detailed study about the use of toric modification and resolutions of non-degenerate complex singularities, see [17]. In section 2, we describe a condition (Proposition 2.4) which implies that f is resolved by the toric modification $P_\Delta \to \mathbf{R}^n$. This condition (called Δ-regularity) is weaker than non-degeneracy. We then present a method drawing pictures of zeros of relatively simple polynomials like the Briançon-Speder's family, and make some claims on topological aspects of real germs defined by them. To discuss the difference between non-degeneracy and Δ-regularity, we present the propagation of regularity in next section. In last two section, we consider the relation between our treatment and equisingular problem.

1. Projective toric varieties and modifications

We recall the construction of the toric variety P_Δ associated with a polyhedron Δ, following to [3] and [15]. However, I do not know any literature concerning the construction of $\widehat{P}_\Delta$ which appeared below.

1.1. **Projective toric varieties.** Let M be the n-dimensional lattice $\mathbf{Z}^n$, and $M^\vee = \mathrm{Hom}(M, \mathbf{Z})$ the dual lattice of M. We identify $(M^\vee)^\vee$ with M in the usual way. We denote by $m = (m_1, ..., m_n)$ an element in $M = \mathbf{Z}^n$, by $a = {}^t(a_1, ..., a_n)$ an element in $M^\vee$, and $\langle a, m\rangle = a(m) = a_1 m_1 + \cdots + a_n m_n$.

Set $K = \mathbf{R}$ or $\mathbf{C}$, and $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x \geq 0\}$. Let Δ be a convex polyhedron in $M_\mathbf{Q} = M \otimes \mathbf{Q}$. We denote $F < \Delta$, if F is a face of Δ. With each face F of Δ we associate a cone C_F in $M_\mathbf{Q}$: to do this we take a point $m \in \mathbf{R}^n$ lying inside the face F, and we set

$$C_F = \mathrm{Cone}(\Delta, F) = \bigcup_{r \geq 0} r \cdot (\Delta - m).$$

Setting

$$\sigma_F = C_F^\vee = \{a \in M_\mathbf{Q}^\vee = M^\vee \otimes \mathbf{Q} \mid \langle a, m\rangle \geq 0, \forall m \in C_F\},$$

the system $\{\sigma_F\}$, as F ranges over the faces of Δ, is a *fan*, which we denote by Σ_Δ. With each face F of Δ, we denote R_F the K-algebra generated by the semi-group $C_F \cap M$. That is,

$$R_F = K[C_F \cap M] = \bigoplus_{m \in C_F \cap M} K x^m.$$

Here, x^m, $m \in M$, are just symbols and the multiplication is defined by $x^{m_1} \cdot x^{m_2} = x^{m_1 + m_2}$. Set $U_F = \mathrm{Spec}(R_F)$ and denote $U_F(K)$ the set of K-valued points of the affine scheme U_F. In other words, $U_F(K)$ is the set of morphisms from $\mathrm{Spec}(K)$ to $\mathrm{Spec}(K[C_F \cap M])$, or equivalently, the set of ring homomorphisms from $K[C_F \cap M]$ to K. Let $m_1, ..., m_p$ be generators of $C_F \cap M$ as a semi-group. Then there is an injection of $U_F(K)$ to K^p defined by $u \mapsto (u(m_1), ..., u(m_p))$. The image of this map has a structure of algebraic varieties. Let $U_F(\mathbf{R}_+)$ be the set of semi-group homomorphisms from $C_F \cap \mathbf{Z}^n$ to $\mathbf{R}_+$. The image of $U_F(\mathbf{R}_+)$ is a semi-algebraic subset, and is homeomorphic to C_F.

If F is a face of F_1, then $C_{F_1}^\vee$ is a face of $C_F^\vee$, thus U_{F_1} (resp. $U_{F_1}(K)$, $U_{F_1}(\mathbf{R}_+)$) is identified with an open subset of U_F (resp. $U_F(K)$, $U_F(\mathbf{R}_+)$). These identifications allow us to glue together of U_F, $U_F(K)$, and $U_F(\mathbf{R}_+)$, as F ranges over the faces of Δ, which are denoted by P_Δ, $P_\Delta(K)$, and $P_\Delta(\mathbf{R}_+)$, respectively. We can start with a fan $\Sigma = \{\sigma_1, ..., \sigma_s\}$ in $M^\vee$, a finite collection of rational polyhedral cones in $M^\vee$ which forms a complex. Gluing $\mathrm{Spec}(K[\sigma_i{}^\vee \cap M])$, $\sigma_i \in \Sigma$, in a similar way, we can construct general toric varieties.

Let F be a face of Δ. A polyhedron Δ is *nonsingular at* F if the σ_F is generated by part of basis of $M^\vee$. A polyhedron is *nonsingular* if it is nonsingular at all faces. If Δ is nonsingular, then $P_\Delta(K)$ is a non-singular variety. We have that $P_\Delta(\mathbf{R}_+)$ is homeomorphic to Δ.

Let Δ_1 be a face of Δ. We set

$$\Delta_1^\perp = \{a \in M^\vee \mid \langle a, \mathrm{Cone}(\Delta_1, F_1)\rangle = 0, \forall F_1 < \Delta_1\},$$

and $M_{\Delta_1} = (\Delta_1^\perp)^\perp$, the minimal sublattice of M containing $\mathrm{Cone}(\Delta_1, F_1) \cap M$, $\forall F_1 < \Delta_1$. For each face F of Δ, we set $I_{\Delta_1, F}$ the ideal in $K[C_F \cap M]$ defined by

$$I_{\Delta_1, F} = \bigoplus_{m \in C_F \cap M - C_F \cap M_{\Delta_1}} K x^m.$$

These ideals $I_{\Delta_1,F}$, $F < \Delta$, define a closed subset in P_Δ (resp. $P_\Delta(K)$, $P_\Delta(\mathbf{R}_+)$) which is canonically isomorphic to P_{Δ_1} (resp. $P_{\Delta_1}(K)$, $P_{\Delta_1}(\mathbf{R}_+)$). We allow a certain freedom in the notation and denote it by the same symbol P_{Δ_1} (resp. $P_{\Delta_1}(K)$, $P_{\Delta_1}(\mathbf{R}_+)$). If F is a face of Δ, then $P_F \subset P_\Delta$ (resp. $P_F(K) \subset P_\Delta(K)$, $P_F(\mathbf{R}_+) \subset P_\Delta(\mathbf{R}_+)$). Set theoretically,

$$P_F \cap P_{F'} = P_{F\cap F'},\ P_F(K) \cap P_{F'}(K) = P_{F\cap F'}(K), \quad \text{and} \quad P_F(\mathbf{R}_+) \cap P_{F'}(\mathbf{R}_+) = P_{F\cap F'}(\mathbf{R}_+).$$

Let $T_F = P_F - \bigcup_{G<F} P_G$, and $T_F(K) = P_F(K) - \bigcup_{G<F} P_G(K)$. Then

$$P_\Delta(K) = \coprod_{F<\Delta} T_F(K) \quad (\text{the } \textit{toric stratification} \text{ of } P_\Delta(K)).$$

For $a \in M^\vee$, we define the derivation

$$\delta_a : K[C_F \cap M] \to K[C_F \cap M]$$

by $x^m \mapsto \langle a, m\rangle x^m$. Here, F is any face of the polyhedron Δ. We understand that δ_a is a vector field on $U_F(K)$, and thus on $P_\Delta(K)$. Since δ_a preserve the ideal $I_{\Delta_1,F}$, $\delta_a|_{\Delta_1} := \delta|_{P_{\Delta_1}(K)}$ gives a tangent vector of $P_{\Delta_1}(K)$, for any $\Delta_1 < \Delta$. The vector fields δ_a, $a \in M^\vee$, generate so-called logarithmic vector fields on $P_\Delta(K)$.

Example 1.1. *Since* $K[\mathbf{Z}^n] = K[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$, *we have* $\delta_{e_i^\vee} = x_i\partial/\partial x_i$, *where* $e_i = (0, ..., 0, \overset{i}{1}, 0, ..., 0)$, $i = 1, ..., n$.

Lemma 1.2. $\delta_a|_{\Delta_1} = 0$, *if and only if* $a \in \Delta_1^\perp$.

Proof. Since $K[C_F \cap M]/I_{\Delta_1,F} = \bigoplus_{m\in C_F\cap M\cap M_{\Delta_1}} Kx^m$, this is obvious. □

1.2. **Projective toric modifications.** Let M_1, M_2 be two lattices, and Δ_1, Δ_2 polyhedra in $M_{1\mathbf{Q}} = M_1 \otimes \mathbf{Q}$, $M_{2\mathbf{Q}} = M_2 \otimes \mathbf{Q}$, respectively. We say Δ_1 *majorizes* Δ_2 if there exist a lattice homomorphism $\alpha : M_2 \to M_1$, and an order preserving map $\beta : \mathrm{sof}(\Delta_1) \to \mathrm{sof}(\Delta_2)$ such that

$$\alpha(\mathrm{Cone}(\Delta_2, \beta(F))) \subset \mathrm{Cone}(\Delta_1, F)$$

for any face F of Δ_1. Here, $\mathrm{sof}(\Delta)$ is the set of faces of Δ. If Δ_1 majorizes Δ_2, then there are canonical maps

$$\beta : P_{\Delta_1} \to P_{\Delta_2}, \quad \beta : P_{\Delta_1}(K) \to P_{\Delta_2}(K), \quad \text{and} \quad \beta : P_{\Delta_1}(\mathbf{R}_+) \to P_{\Delta_2}(\mathbf{R}_+),$$

induced by the natural homomorphism of K-algebras

$$K[\mathrm{Cone}(\Delta_2, \beta(F)) \cap M_2] \to K[\mathrm{Cone}(\Delta_1, F) \cap M_1].$$

Lemma 1.3. *Under the above notation, we have that* $d\beta(\delta_a) = \delta_{\alpha^\vee(a)}$, *where* $\alpha^\vee$ *is the dual morphism of* α.

Proof. It is enough to show the case $\Delta_1 = M_{1\mathbf{Q}}$, and $\Delta_2 = M_{2\mathbf{Q}}$. Set $M_1 = \mathbf{Z}^{n_1}$ and $M_2 = \mathbf{Z}^{n_2}$. For $e_i \in M_2$ with $i = 1, ..., n_2$, we write $\alpha(e_i) = (a_i^1, ..., a_i^{n_1})$. If we identify $K[M_1], K[M_2]$ with $K[y_1, y_1^{-1}, ..., y_{n_1}, y_{n_1}^{-1}]$, $K[x_1, x_1^{-1}, ..., x_{n_2}, x_{n_2}^{-1}]$, respectively, we have that $x_i \circ \beta(y) = y_1^{a_i^1} \cdots y_{n_1}^{a_i^{n_1}}$, for $i = 1, ..., n_2$. By routine calculation, we obtain

$$d\beta\left(y_j \frac{\partial}{\partial y_j}\right) = \sum_{i=1}^{n_2} a_i^j x_i \frac{\partial}{\partial x_i},$$

which proves the lemma. □

Example 1.4. *Let* Δ_1 *be a trapezoid* $A_1A_2B_1B_2$ *so that the segment* A_1B_1 *is parallel to* A_2B_2. *Let* Δ_2 *be a segment* AB. *Then* Δ_1 *majorizes* Δ_2 *by the map defined by* $A_i \mapsto A, B_i \mapsto B, i = 1, 2$. *This gives a* KP^1*-bundle* $P_{\Delta_1}(K) \to P_{\Delta_2}(K) = KP^1$.

For a convex polyhedron Δ in $\mathbf{R}^n$ majorizing $\mathbf{R}^n_+$, we get maps

$$\rho_\Delta : P_\Delta(K) \to P_{\mathbf{R}^n_+}(K) = K^n,$$

which we call the *projective toric modification* of K^n defined by Δ.

We continue the notation above. We set $m = 1 + \sum_{i=1}^n e_i 2^{i-1}$, for $e_i \in \{0, 1\}$, and

$$A_m = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid \operatorname{sign} x_i = (-1)^{e_i}\}.$$

We denote by $A_m(\Delta)$ the closure of $\rho_\Delta^{-1}(A_m)$ in $P_\Delta(K)$. Each $A_m(\Delta)$ is homeomorphic to Δ, and

$$P_\Delta(\mathbf{R}) = \bigcup_{1 \le m \le 2^n} A_m(\Delta).$$

Remark that P_Δ is obtained by gluing of $A_m(\Delta)$'s along all faces of Δ. Set $\widehat{P}_\Delta$ the space obtained by gluing of $A_m(\Delta)$ along the faces of Δ in the coordinate planes, and $\widehat{\rho}_\Delta$ the natural map of $\widehat{P}_\Delta$ to $\mathbf{R}^n$. We remark that there is a natural map $p_\Delta : \widehat{P}_\Delta \to P_\Delta(\mathbf{R})$ such that $\rho_\Delta \circ p_\Delta = \widehat{\rho}_\Delta$.

Example 1.5. *Let Δ be a convex polyhedron in $\mathbf{R}^n$ coinciding with $\mathbf{R}^n_+$ outside some compact set. Then Δ majorizes $\mathbf{R}^n_+$ and we get maps*

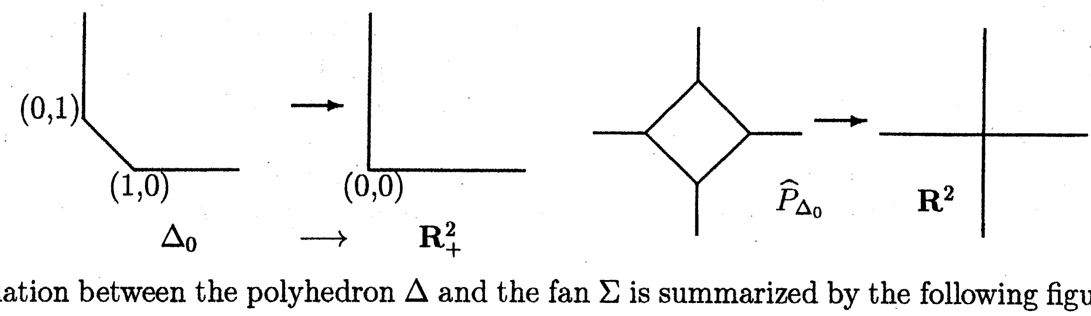
$$\begin{aligned} \rho_\Delta : & P_\Delta(K) \to P_{\mathbf{R}^n_+}(K) = K^n, \quad K = \mathbf{R}, \mathbf{C}, \\ \rho_{\Delta,+} : & P_\Delta(\mathbf{R}_+) \to P_{\mathbf{R}^n_+}(\mathbf{R}_+) = \mathbf{R}^n_+, \quad \text{and} \quad \widehat{\rho}_\Delta : \widehat{P}_\Delta \to \mathbf{R}^n. \end{aligned}$$

We have that ρ_Δ is proper and is an isomorphism over $K^n - \{0\}$. The exceptional set $\rho^{-1}(0)$ consists of the varieties P_F, where F ranges over the compact faces of Δ.

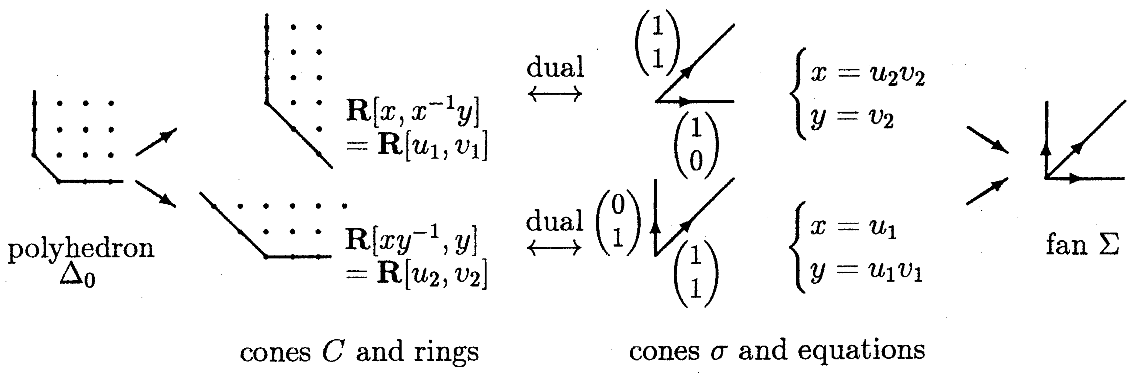
Definition 1.6 ([7] p.221). *Let $n \ge 2$ and S be the unit sphere with center at the origin in $\mathbf{R}^n$. Let $\pi_1 : \mathbf{R} \times S \to \mathbf{R}^n$ by $(t, v) \mapsto tv$. This is a degree two proper map of real analytic manifolds, which is called double oriented blowing up of $\mathbf{R}^n$. The map π_1 induces $\pi_2 : \overline{X} = \mathbf{R}_+ \times S \to \mathbf{R}^n$, which is called (simple) oriented blowing up . It also induces $\pi_3 : X = \mathbf{R} \times S/\mathbf{Z}_2 \to \mathbf{R}^n$, where $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z} = \{\pm 1\}$ acts on $\mathbf{R} \times S$ by $(t, v) \mapsto (-t, -v)$. This π_3 is called the (non-oriented) blowing up of $\mathbf{R}^n$ with center $0 \in \mathbf{R}^n$.*

Set $\Delta_0 = \{(\nu_1, \dots, \nu_n) \in \mathbf{R}^n_+ \mid \nu_1 + \dots + \nu_n \ge 1\}$. Then ρ_{Δ_0} is the blowing up of $\mathbf{R}^n$ with center $0 \in \mathbf{R}^n$. Moreover, $\widehat{\rho}_{\Delta_0}$ is homeomorphic to the oriented blowing up of $\mathbf{R}^n$ with center $0 \in \mathbf{R}^n$. Thus we may understand that the map $\widehat{P}_\Delta \to \mathbf{R}^n$ is, at least topologically, a toric analogue of the (simple) oriented blow-up.

The following figures explain the blowing up of $\mathbf{R}^2$ at the origin.



The relation between the polyhedron Δ and the fan Σ is summarized by the following figure.



For an $a \in M^\vee$, and a polyhedron Δ in M, we set

$$\ell_\Delta(a) = \min\{\langle a, m\rangle \mid m \in \Delta\}, \quad \text{and} \quad \gamma_\Delta(a) = \{m \in \Delta \mid \langle a, m\rangle = \ell_\Delta(a)\}.$$

Let Δ_1, Δ_2 be nonsingular polyhedra in $M_{\mathbf{Q}}$, such that Δ_1 majorizes Δ_2. Set $\mathrm{sov}(\Delta_1/\Delta_2)$ be the set of primitive $a \in M^\vee$, with $\dim \gamma_{\Delta_1}(a) = n-1$ and $\dim \gamma_{\Delta_2}(a) < n-1$. $\mathrm{sof}(\Delta_1/\Delta_2)$ denotes the set of faces of the form $\bigcap_{a\in A} \gamma_{\Delta_1}(a)$, $A \subset \mathrm{sov}(\Delta_1/\Delta_2)$. Since $P_{\gamma(a)}(K)$, $a \in \mathrm{sov}(\Delta_1/\Delta_2)$, are exceptional divisors for $P_{\Delta_1}(K) \to P_{\Delta_2}(K)$, the system

$$\left\{ E_F(K) := P_F(K) - \bigcup_{G<F, G\in \mathrm{sof}(\Delta_1/\Delta_2)} P_G(K) \right\}, \quad F \in \mathrm{sof}(\Delta_1/\Delta_2)$$

gives the *exceptional stratification* for $P_{\Delta_1}(K) \to P_{\Delta_2}(K)$.

2. Resolution and Newton polyhedron

Let $f(x)$ be an analytic function of n variables $x = (x_1, \ldots, x_n)$, defined in a neighborhood of the origin of K^n. In this section, we consider when a toric modification, constructed in the preceding section, gives a resolution of the function of $f(x)$. Set $\mathrm{grad}\, f$ the gradient vector of f

$$\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

We start the case of the simplest toric modification: the blowing up at the origin. Let $\beta : M(K) \to K^n$ be the blowing up at the origin, and $E(K) = \beta^{-1}(0) \simeq KP^{n-1}$, the exceptional set of β. Let $[f]$ be the initial polynomial of f. Then the following proposition is well-known.

Proposition 2.1. *The following statements are equivalent.*

(i) *The strict transform of f by β is nonsingular and β gives a resolution of f.*
(ii) $\{\mathrm{grad}[f] = 0\} \subset \{(x_1, \ldots, x_n) = 0\}$.
(iii) *$[f]$ defines a nonsingular variety in the projective space KP^{n-1}.*

Let us briefly consider how our principle works in this case. Suppose that f satisfies the conditions above, and $n = 3$, $K = \mathbf{R}$. According to our principle, if we can understand the location of the zero locus of $[f]$ in KP^2, we can draw a picture of $f^{-1}(0)$ near the origin. If the degree of $[f]$ is low, then we can determine possible pictures, using the list of topological classification of non-singular real plane curves. There exist a complete list of non-singular real plane curve with degree ≤ 7. See the works of D.A.Gudkov ([7]) and O.Ya.Viro ([19]). If the degree of $[f]$ becomes high, this is a very hard problem (the Hilbert's 16th problem).

Let

$$\sum_\nu c_\nu x^\nu = \sum_\nu c_{\nu_1,\dots,\nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n}, \quad \nu = (\nu_1, \dots, \nu_n)$$

be the Taylor expansion of $f(x)$ at the origin. Let $\Gamma_+(f)$ be the *Newton polyhedron* of f at the origin, that is, the convex hull in $\mathbf{R}^n$ of the set

$$\{\nu + \mathbf{R}_+^n \mid c_\nu \neq 0\}.$$

Let Δ be a nonsingular polyhedron in $\mathbf{R}^n = M \otimes \mathbf{R}$ majorizing $\mathbf{R}_+^n$. The first problem we are facing is when $P_\Delta \to K^n$ gives a resolution of f. In this section, we seek the condition for f such that $P_\Delta \to K^n$ is a resolution of f.

For a face F of Δ, we take a $Q_F \in M$ satisfying

$$\Gamma_+(f) \subset Q_F + \mathrm{Cone}(\Delta, F), \quad \text{and}$$
$$\Gamma_+(f) \not\subset Q_F + \mathrm{Cone}(\Delta, F) + \mathbf{N}^n.$$

We set

$$f_\gamma = \sum_{\nu \in \gamma} c_\nu x^\nu \quad \text{where } \gamma = \gamma(F) = (Q_F + M_F) \cap \Gamma_+(f).$$

We then have the canonical morphism $T_F(K) \to T_\gamma(K)$, induced by the embedding $M_\gamma \hookrightarrow MF$, whose fiber is $(K - \{0\})^{\dim F - \dim \gamma}$.

Let $Z = Z_\Delta(f)$ (resp. $\widehat{Z} = \widehat{Z}_\Delta(f)$) be the zero locus of the proper transform of f via ρ_Δ (resp. $\widehat{\rho}_\Delta$). Remark that $Z_\Delta(f)$ contains the proper transform of $f^{-1}(0)$, but does not equal in general. Then we have the following lemmas.

Lemma 2.2. (i) *If $\gamma(F)$ is not empty, then*

$$Z_\Delta(f) \cap T_F(K) \cong E_\gamma(f) \times (K - \{0\})^{\dim F - \dim \gamma},$$

where $E_\gamma(f)$ is the algebraic set defined by $f_\gamma = 0$ in $T_\gamma(K)$.

(ii) *If $\gamma(F)$ is empty, then $Z_\Delta(f) \cap T_F(K) = T_F(K)$.*

Proof. By nonsingularity of Δ, $K[C_P \cap M]$ is isomorphic to $K[y_1, \dots, y_n]$ for a vertex P of the face F. We note that x_i can be written in the form $y_1^{a_i^1} \cdots y_n^{a_i^n}$, for $i = 1, \dots, n$. Set $a^j = {}^t(a_1^j, \dots, a_n^j)$ and $\ell_j = \ell_{\Gamma_+(f)}(a^j)$. We may assume that $\bigcap_{j=s+1}^n \gamma_{\Gamma_+(f)}(a^j) = \gamma$. Then the lift of f to $K[C_F \cap M]$ can be written in the following form:

$$f(y_1^{a_1^1} \cdots y_n^{a_1^n}, \dots, y_1^{a_n^1} \cdots y_n^{a_n^n}) = y_1^{\ell_1} \cdots y_n^{\ell_n} f_P(y_1, \dots, y_n).$$

Here we have

$$\begin{aligned} f_P|_{T_F(K)}(y_1, \dots, y_s) &= y_1^{-\ell_1} \cdots y_n^{-\ell_n} f_\gamma(y_1^{a_1^1} \cdots y_n^{a_1^n}, \dots, y_1^{a_n^1} \cdots y_n^{a_n^n}) \\ &= y_1^{-\ell_1} \cdots y_s^{-\ell_s} f_\gamma(y_1^{a_1^1} \cdots y_s^{a_1^s}, \dots, y_1^{a_n^1} \cdots y_s^{a_n^s}). \end{aligned}$$

This implies the lemma. □

It is not difficult to modify the proof of the lemma above for the case a polyhedron Δ which is not nonsingular. We also remark that f and $x^m f$ have same zeros in $T_F(K)$, for $m \in M$, and a polynomial f.

Lemma 2.3. *The following statements are equivalent.*

(i) *Z is nonsingular near $T_F(K)$ and intersects transversely with $T_F(K)$.*

(ii) $\{\mathrm{grad}\, f_\gamma = 0\} \subset \{x_1 \cdots x_n = 0\}$.

We say $f(x)$ is *non-degenerate* if grad f_γ is not zero except $\{x_1 \cdots x_n = 0\}$ for any compact face γ of $\Gamma_+(f)$. By the lemma above, we have $f(x)$ is non-degenerate if and only if $Z_\Delta(f)$ is nonsingular and intersects transversely with the toric stratification of $P_\Delta(K)$ near $\rho^{-1}(0)$, for any nonsingular polyhedron Δ majorizing $\Gamma_+(f)$.

For a face F of a polyhedron Δ we denote by $S(F)$ the set

$$\left\{ \prod_{i \in I(P)} x_i = 0, \quad \text{for any vertex } P \text{ of } F \right\},$$

where $I(P)$ is the set of numbers $i \in \{1, ..., n\}$ with $\langle e_i^\vee, P\rangle > 0$.

Proposition 2.4. *For $F \in \mathrm{sof}(\Delta/\mathbf{R}^n_+)$, the following statements are equivalent.*

(i) *Z is nonsingular near $E_F(K)$, and intersects transversely with $E_F(K)$.*

(ii) *$\{\mathrm{grad}\, f_\gamma = 0\} \subset S(F)$.*

We say f is Δ*-regular* if the condition (ii) is satisfied for any compact face $F \in \mathrm{sof}(\Delta/\mathbf{R}^n_+)$.

Proof. Since Δ is nonsingular, $K[C_P \cap M]$ is isomorphic to $K[y_1, ..., y_n]$ for a vertex P of the face F. Remember that x_i can be written in the form $y_1^{a_i^1} \cdots y_n^{a_i^n}$, for $i = 1, ..., n$. Set $a^j = {}^t(a_1^j, ..., a_n^j)$. We may assume that $I(P) = \{t+1, ..., n\}$, $a^j = e_j^\vee$, $j = 1, ..., t$, and $\bigcap_{j=s+1}^p \gamma_\Delta(a^j) = F$. We thus have

$$x_i = \begin{cases} y_i y_{t+1}^{a_i^{t+1}} \cdots y_n^{a_i^n}, & \text{for } i = 1, ..., t, \\ y_{t+1}^{a_i^{t+1}} \cdots y_n^{a_i^n}, & \text{for } i = t+1, ..., n. \end{cases}$$

Setting $\ell_j = \ell_{\Gamma_+(f)}(a_j)$, and $\gamma = \gamma(F)$, we define $f_P(y)$, $f_{\gamma,P}(y)$, and $f_{\gamma,P}(\tilde{y})$, by

$$\begin{aligned} f(x) &= y_{t+1}^{\ell_{t+1}} \cdots y_n^{\ell_n} f_P(y), \\ f_\gamma(x) &= y_{t+1}^{\ell_{t+1}} \cdots y_n^{\ell_n} f_{\gamma,P}(y), \quad \text{and} \\ f_\gamma(\tilde{x}) &= \tilde{y}_{t+1}^{\ell_{t+1}} \cdots \tilde{y}_n^{\ell_n} f_{\gamma,P}(\tilde{y}). \end{aligned}$$

Here,

$$\begin{aligned} \tilde{x} &= (\tilde{x}_1, ..., \tilde{x}_n), \quad \tilde{x}_i = \tilde{y}_1^{a_i^1} \cdots \tilde{y}_n^{a_i^n}, \\ \tilde{y} &= (\tilde{y}_1, ..., \tilde{y}_n), \quad \tilde{y}_j = \begin{cases} 1, & \text{if} \quad j = s+1, ..., p, \\ y_j, & \text{if} \quad j = 1, ..., s, p+1, ..., n. \end{cases} \end{aligned}$$

Remark that $f(x)$ is divisible by $x_j^{\ell_j}$, for $j = 1, ..., t$. We thus have

$$\begin{aligned} \frac{\partial f}{\partial y_j}(x) &= y_{t+1}^{\ell_{t+1}} \cdots y_n^{\ell_n} \left(\ell_j \frac{f_P(y)}{y_j} + \frac{\partial f_P}{\partial y_j}(y) \right), \quad j = 1, ..., t, \\ y_j \frac{\partial f}{\partial y_j}(x) &= y_{t+1}^{\ell_{t+1}} \cdots y_n^{\ell_n} \left(\ell_j f_P(y) + y_j \frac{\partial f_P}{\partial y_j}(y) \right), \quad j = t+1, ..., n, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f_\gamma}{\partial y_j}(x) &= y_{t+1}^{\ell_{t+1}} \cdots y_n^{\ell_n} \left(\ell_j \frac{f_{\gamma,P}(y)}{y_j} + \frac{\partial f_{\gamma,P}}{\partial y_j}(y) \right), \quad j = 1, ..., t, \\ y_j \frac{\partial f_\gamma}{\partial y_j}(x) &= y_{t+1}^{\ell_{t+1}} \cdots y_n^{\ell_n} \left(\ell_j f_{\gamma,P}(y) + y_j \frac{\partial f_{\gamma,P}}{\partial y_j}(y) \right), \quad j = t+1, ..., n. \end{aligned}$$

We also have

$$\frac{\partial f_\gamma}{\partial y_j}(\tilde{x}) = \tilde{y}_{t+1}^{\ell_{t+1}} \cdots \tilde{y}_n^{\ell_n} \left(\ell_j \frac{f_{\gamma,P}(\tilde{y})}{\tilde{y}_j} + \frac{\partial f_{\gamma,P}}{\partial y_j}(\tilde{y}) \right), \quad j = 1, ..., t,$$

$$y_j \frac{\partial f_\gamma}{\partial y_j}(\tilde{x}) = \tilde{y}_{t+1}^{\ell_{t+1}} \cdots \tilde{y}_n^{\ell_n} \left(\ell_j f_{\gamma,P}(\tilde{y}) + \tilde{y}_j \frac{\partial f_{\gamma,P}}{\partial y_j}(\tilde{y}) \right), \quad j = t+1, ..., n.$$

Since $y_j \frac{\partial}{\partial y_j} = \sum_{i=1}^n a_i^j x_i \frac{\partial}{\partial x_i}$, the following hold.

$$\frac{\partial f_\gamma}{\partial y_j} = y_{t+1}^{a_1^{t+1}} \cdots y_n^{a_1^n} \frac{\partial f_\gamma}{\partial x_j}, \quad j = 1, ..., t,$$

$$y_j \frac{\partial f_\gamma}{\partial y_j} = \sum_{i=1}^n a_i^j x_i \frac{\partial f_\gamma}{\partial x_i}, \quad j = t+1, ..., n,$$

$$y_j \frac{\partial f_\gamma}{\partial y_j} = \sum_{i=1}^n a_i^j x_i \frac{\partial f_\gamma}{\partial x_i} = \ell_j f_\gamma, \quad j = s+1, ..., p.$$

Therefore we obtain that $Z_\Delta(f)$ is nonsingular and intersect transversely with $E_F(K)$ at y

$$\Leftrightarrow \quad y \notin \left\{ f_P(y) = 0, \frac{\partial f_P}{\partial y_j}(y) = 0, j = 1, ..., s, p+1, ..., n \right\}$$

$$\Leftrightarrow \quad \tilde{y} \notin \left\{ f_{\gamma,P}(\tilde{y}) = 0, \frac{\partial f_{\gamma,P}}{\partial y_j}(\tilde{y}) = 0, j = 1, ..., s, p+1, ..., n \right\}$$

$$\Leftrightarrow \quad \tilde{y} \notin \left\{ \frac{\partial f_\gamma}{\partial x_i}(\tilde{y}_1^{a_1^1} \cdots \tilde{y}_n^{a_1^n}, ..., \tilde{y}_1^{a_n^1} \cdots \tilde{y}_n^{a_n^n}) = 0, i = 1, ..., n \right\}$$

$$\Leftrightarrow \quad \tilde{x} \notin \left\{ \frac{\partial f_\gamma}{\partial x_i}(\tilde{x}) = 0, i = 1, ..., n \right\}.$$

Since $\prod_{i \in I(P)} \tilde{x}_i \neq 0$, if $y \in E_F(K)$, this completes the proof of the lemma. $\square$

3. Real plane curve germ

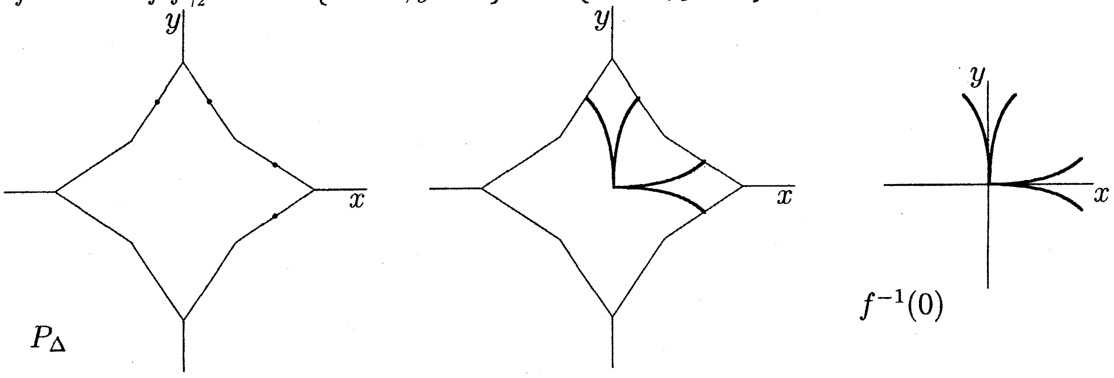
In this section, we consider function-germs $f = f(x,y)$ defined by polynomials in variables (x,y). Let $f(x,y)$ be a non-degenerate function. For simplicity we assume that f is not divisible by x and y. Then the Newton polygon $\Gamma_+(f)$ intersects each axis. For a compact face γ of $\Gamma_+(f)$, $f_\gamma(x,y)$ is a weighted homogeneous polynomial and the zero locus of f_γ is invariant under the $\mathbf{R}^*$-action defined by $(x,y) \mapsto (t^p x, t^q y)$, $t \in \mathbf{R}^*$. Here (p,q) is the vector supporting the face γ of $\Gamma_+(f)$. Let $n_{++}(\gamma)$ (resp. $n_{+-}(\gamma)$, $n_{-+}(\gamma)$, $n_{--}(\gamma)$) be the number of the half-branches of $\{f_\gamma = 0\}$ in the region $\{x > 0, y > 0\}$ (resp. $\{x > 0, y < 0\}$, $\{x < 0, y > 0\}$, $\{x < 0, y < 0\}$).

Let $\Gamma_+(f)$ be an integral polyhedron in the first quadrant. We draw the images of $\Gamma_+(f)$ by $(x,y) \mapsto (\pm x, \pm y)$ in all four quadrants. Take $n_{++}(\gamma)$ points in γ and connect these points with the origin by a curve which is close to the corresponding half branch of $f_\gamma = 0$ in $\{x > 0, y > 0\}$ near the origin. We can draw half-branches similarly in each quadrant. Drawing all half-branches in all quadrants in this way, we obtain the local picture of $f = 0$ near the origin.

This can be understood in the following way: We set $\Delta = \Gamma_+(f)$, or a regular polyhedron majorizing $\Gamma_+(f)$. Then $\widehat{P}_\Delta$ is obtained by gluing of four copies of Δ along non-compact faces of Δ. If $f(x,y)$ is non-degenerate, the set $(\mathbf{R}^2, f^{-1}(0), 0)$ can be resolved by a toric modification defined by Δ. The number of components of the strict transforms of f which intersect to the exceptional set corresponds to a compact face γ is described by the numbers

$n_{\pm\pm}(\gamma)$. If we know these numbers it is not difficult to draw the picture of zeros of f in $\widehat{P}_\Delta$. So the image of this set by $\widehat{P}_\Delta \to \mathbf{R}^2$ can be draw as in the previous paragraph.

Example 3.1. *Let $f(x,y) = x^2y^2 - x^5 - y^5$. Let Δ denote its Newton polyhedron. We denote by γ_1 the segment connecting $(2,2)$ and $(5,0)$, and by γ_2 the segment connecting $(2,2)$ and $(0,5)$. Then $f_{\gamma_1} = x^2y^2 - x^5 = x^2(y^2 - x^3)$ and there is one half-branch of $f_{\gamma_1} = 0$ in $\{x > 0, y > 0\}$ and $\{x > 0, y < 0\}$. Similarly $f_{\gamma_2} = x^2y^2 - y^5 = y^2(x^2 - y^3)$ and there is one half-branch of $f_{\gamma_2} = 0$ in $\{x > 0, y > 0\}$ and $\{x < 0, y > 0\}$.*



I believe many people know the result in this section (at least implicitly). But I do not know any article which presents this explicitly.

4. Real surface germ in $\mathbf{R}^3$

In this section, we consider function-germs $f = f(x,y,z)$ defined by polynomials in variables (x,y,z). We set $\Delta = \Gamma_+(f)$ or a regular polyhedron majorizing $\Gamma_+(f)$. Then $\widehat{P}_\Delta$ is obtained by gluing of eight copies of Δ along non-compact faces of Δ.

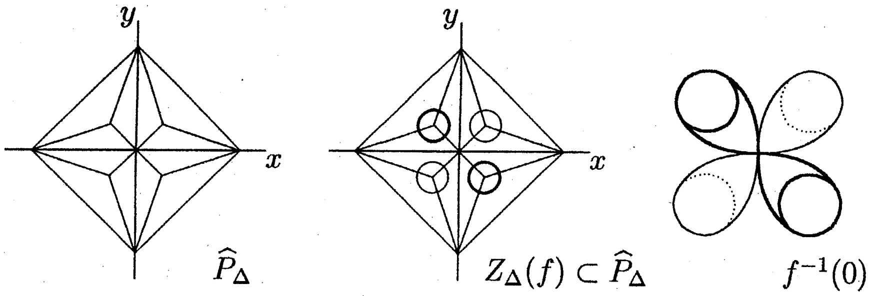
Let F be a face of Δ. We assume $F = \gamma$ when $\Delta = \Gamma_+(f)$, and F lies over γ otherwise. Let P_F be the corresponding component of the exceptional locus of $P_\Delta \to \mathbf{R}^n$. Let $\widehat{P}_F$ be the corresponding set in $\widehat{P}_\Delta$. We denote by $Z_F(f)$ (resp. $\widehat{Z}_F(f)$) the intersection of the strict transform of the zero of f and P_F (resp. $\widehat{P}_F$). This is determined by f_γ. If we draw a picture of $\widehat{Z}_F(f)$ in $\widehat{P}_F$, we are able to draw a picture of $\{f(x,y,z) = 0\}$ near the origin. For example, if Δ coincides the positive orthant except some compact set, an approximate picture of the zero set of f is obtained by taking a cone of

$$\left(\bigcup_{F:\text{compact}} \widehat{P}_F, \bigcup_{F:\text{compact}} Z_F(f) \right)$$

whose vertex is the origin.

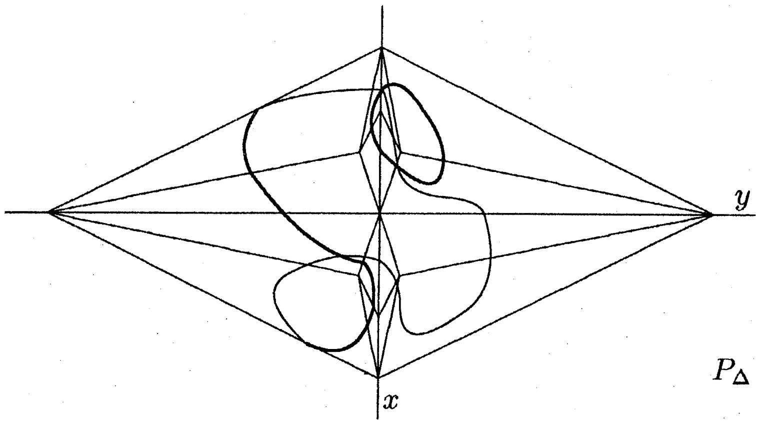
We say that f is a *trigonal trinomial*, if each 2-dimensional face of the Newton polyhedron $\Gamma_+(f)$ of f is a simplex, i.e., a triangle, and f_γ is a trinomial for any 2-dimensional face γ of $\Gamma_+(f)$. Let $F = \gamma$ be a 2-face of $\Gamma_+(f)$. Then $\widehat{P}_F$ is eight copies of triangles and f_γ is trinomial. This it is easy to draw the picture of $\widehat{Z}_F(f)$ in $\widehat{P}_F$. Therefore it is easy to draw a local picture of zeros of trigonal trinomials near the origin.

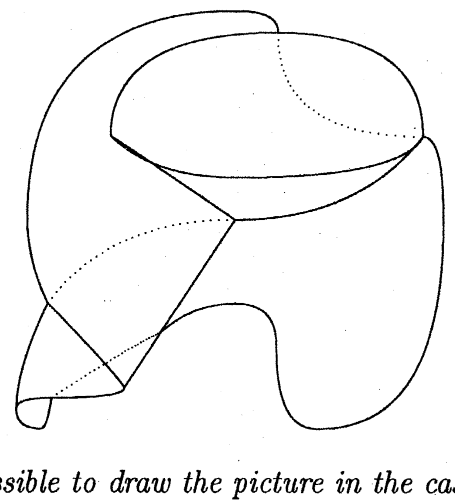
Example 4.1 ($T_{p,q,r}$). *Let $f(x,y,z) = x^p + y^q + z^r + axyz$. This is a trigonal polynomial and we can draw the picture of the zero of f near 0. When $p = q = r = 4$ and $a = 1$, it looks like the following:*



Thick lines correspond the components in $\{z \geq 0\}$ and thin lines correspond the components in $\{z \leq 0\}$. All these pictures are viewed from the point $(0, 0, \infty)$.

Example 4.2 ([16]). *Let $f(x, y, z) = x^8 + y^\ell + z^\ell + tx^5z^2 + x^3yz^3$, $\ell \geq 16$. This is a μ-constant family. This is a trigonal polynomial when $\ell \geq 17$ and we can draw the picture of the zero of f near 0. Let Δ denote the Newton polygon of f_t, $t \neq 0$. When $\ell = 17$ and $t = 1$, the picture in $\widehat{P}_\Delta$ looks like the following picture. Here thick lines correspond to the zero locus in $\{z \geq 0\}$ and thin lines correspond that in $\{z \leq 0\}$.*

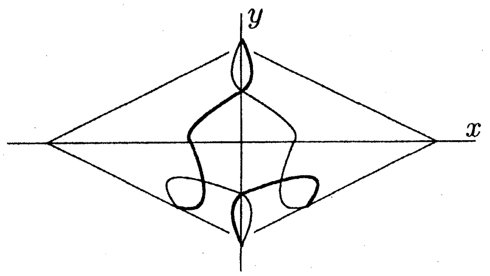


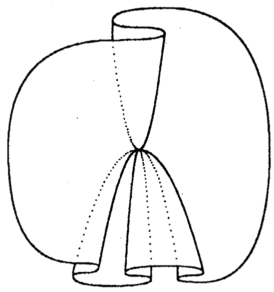


$f^{-1}(0)$

In a similar way, it is possible to draw the picture in the case $t < 0$. I hope the reader to enjoy investigating the difference between the cases $t > 0$ and $t < 0$. I also left the reader to draw the pictures for the other ℓ's.

Example 4.3 ([2]). $f(x, y, z) = z^5 + ty^6z + xy^7 + x^{15}$. *When $t \neq 0$, the Newton polyhedron of f contains a quadrangle as a face. Let γ denote the compact 2-face of $\Gamma_+(f)$. defined by $(x, y, z) \mapsto (x, y)$ to the zero of f_γ. We also work about the discriminant of the restriction of the projection Then we obtain that the picture of $\{f(x, y, z) = 0\}$ with $t = -1$ looks like the following:*





Next we consider the case $t \geq 0$. Drawing the picture in a similar way, we obtain that the restriction of the projection $(x,y,z) \mapsto (x,y)$ to $\{f(x,y,z)=0\}$ is a homeomorphism.

It is reasonable to expect that Newton polyhedron of f gives some restrictions for the topology of $(\mathbf{R}^3, f^{-1}(0), 0)$ when f is non-degenerate. In the remaining of the section, we consider a restriction on the Euler characteristics of local level surface of f.

For each 2-face F of $\Gamma_+(f)$, we denote by $v_2(F)$ the area of F and $v_1(F)$ denotes the perimeter of F. We also denote by $e(F)$ the number of 1-faces of F. We set

$$A = \sum_F \left(\frac{3}{4}v_2(F) - \frac{1}{4}v_1(F) + 1\right), \quad \text{and} \quad A_1 = \sum_F (4 - e(F))$$

where the summations are taken over all compact 2-faces of $\Gamma_+(f)$. For each vertex u of $\Gamma_+(f)$ we denote by n_u the number of 1-faces containing u. We define the number B by

$$B = \sum_u (2^{i(u)-1} - n_u)$$

where the summation is taken over all vertices u of $\Gamma_+(f)$ such that the coefficient of x^u in the Taylor expansion of f at 0 is positive. Here we denote by $i(u)$ the number of non-zero components of u.

Theorem 4.4. *Let $\chi_+(f)$ denote the Euler characteristic of the local positive level set*

$$\{x \in \mathbf{R}^3 \mid f(x) = \delta,\ |x| < \varepsilon\}, \quad \textit{for} \quad 0 < \delta \ll \varepsilon \ll 1.$$

We assume that $\Gamma_+(f)$ is even, i.e., twice of some integral polyhedron. If $f(x,y,z)$ is a non-degenerate function, then we have the following inequality:

$$-2A + 2A_1 + 2B \leq \chi_+(f) \leq 2A + 2B.$$

Proof. We set $\Delta = \Gamma_+(f)$ and consider maps $\rho_\Delta : P_\Delta \to \mathbf{R}^3$, and $\widehat{\rho}_\Delta : \widehat{P}_\Delta \to \mathbf{R}^3$. Let F be a 2-face of Δ. Let $P_F(f \geq 0)$ denote the intersection of the closure of $\rho_\Delta^{-1}\{f > 0\}$ and P_F, and $\widehat{P}_F(f \geq 0)$ the intersection of the closure of $\widehat{\rho}_\Delta^{-1}\{f > 0\}$ and $\widehat{P}_F$. Let u be a vertex of Δ such that the coefficient of x^u in the Taylor expansion of f at 0 is positive. We first observe that

$$\lim_{\delta \to +0} \{x \mid f(x) = \delta,\ |x| < \varepsilon\} = \bigcup_F \widehat{P}_F(f \geq 0).$$

Since $P_\Delta \to \mathbf{R}^3$ gives a resolution of $\{f = 0\}$, we obtain that

$$\chi_+(f) = 2\chi\left(\bigcup_{F:\text{compact 2-face}} \widehat{P}_F(f \geq 0)\right).$$

We denote by $\widehat{B}_u$ the union of one eighth's of small "ball" centered at the points corresponds to u in $\widehat{P}_\Delta$ such that $\widehat{P}_F \cap \widehat{B}_u \subset \widehat{P}_F(f \geq 0)$. Then we have

$$\begin{aligned}
&\sum_F 2\chi\left(\bigcup_{F:\text{compact 2-face}} \widehat{P}_F(f \geq 0)\right) \\
&= \sum_F 2\chi\left(\widehat{P}_F(f \geq 0) - \bigcup_u \widehat{B}_u \cap P_F(f \geq 0)\right) + \sum_F \sum_{u \in F} \chi\left(B_u \cap \widehat{P}_F(f \geq 0)\right) \\
&= \sum_F 2\chi\left(\widehat{P}_F(f \geq 0) - \bigcup_u \widehat{B}_u \cap \widehat{P}_F(f \geq 0)\right) + \sum_u 2^{i(u)}
\end{aligned}$$

We denote by B_u a small ball centered at the point corresponds to u in P_Δ such that $P_F \cap B_u \subset Z_F(f \geq 0)$. Since we have

$$\chi(P_F(f \geq 0)) = \chi\left(P_F(f \geq 0) - \bigcup_u B_u \cap P_F(f \geq 0)\right) + \sum_{u \in F} \chi(B_u \cap P_F(f \geq 0)),$$

we obtain that

$$\sum_F \chi(P_F(f \geq 0)) = \sum_F \chi\left(P_F(f \geq 0) - \bigcup_u B_u \cap P_F(f \geq 0)\right) + \sum_u n_u.$$

Using the equality

$$\sum_F \chi\left(P_F(f \geq 0) - \bigcup_u B_u \cap P_F(f \geq 0)\right) = \sum_F \chi\left(\widehat{P}_F(f \geq 0) - \bigcup_u \widehat{B}_u \cap \widehat{P}_F(f \geq 0)\right),$$

we thus obtain that

$$\chi_+(f) - \sum_u 2^{i(u)} = 2\sum_F \chi(P_F(f \geq 0)) - 2\sum_u n_u.$$

By Theorem 0.3 of [5], we have

$$3 - e(F) + \frac{1}{4}v_1(F) - \frac{3}{4}v_2(F) \leq \chi(P_F(f \geq 0)) \leq 1 - \frac{1}{4}v_1(F) + \frac{3}{4}v_2(F),$$

and we thus have $A_1 - A \leq \sum_F \chi(P_F(f \geq 0)) \leq A$. This completes the proof. □

5. Propagation of regularity

Let Δ be a polyhedron. We consider a face of Δ with the following properties:

- F is of codimension m.
- There exist integral vectors a^1, ..., a^m with

$$F = \gamma_\Delta(a^1) \cap \cdots \cap \gamma_\Delta(a^m).$$

We next consider a subset I of $\{1, ..., n\}$ with the following property:

- $F_I := F \cap \bigcap_{i \in I} \gamma_\Delta(e^i)$ is of codimension $m + \#I$.

We set

$$S_{I,i} := \{(m_1, ..., m_n) \in \mathbf{R}^n \mid m_i = 1, m_j = 0 \text{ for } j \in I - \{i\}\}.$$

Proposition 5.1. *We assume that T_F is in the regular locus of P_Δ. Then the following conditions are equivalent.*

(i) *T_{F_I} is in the regular locus of P_Δ.*
(ii) *$(F^\perp)^\perp \cap S_{I,i} \cap \mathbf{Z}^n \neq \emptyset$ for any $i \in I$.*

This lemma is a generalization of a trick appeared in [16] which treats the case $n=3$.

Proof. Without loss of generality, we may assume $I=\{s+1,...,n\}$. Then the condition (i) is equivalent that

$$\det(a^1,...,a^m,e^{s+1},...,e^n)=1, \quad \text{i.e.} \quad \det(\tilde{a}^1,...,\tilde{a}^m)=1,$$

where $\tilde{a}^j={}^t(a_1^j,...,a_s^j)$, $j=1,...,m$. By the equivalence of (i) and (viii) in the following lemma, this is equivalent to the condition (ii). $\square$

Lemma 5.2. *Let $a^j={}^t(a_1^j,...,a_n^j)$, $j=1,...,m$, be integral vectors. Let s be an integer with $m\leq s\leq n$. We set $\tilde{a}^j={}^t(a_1^j,...,a_s^j)$, and $\overline{a}^j={}^t(a_{s+1}^j,...,a_n^j)$. We assume that $\det(a^1,...,a^m)=1$. Then the following conditions are equivalent:*

(i) $\det(\tilde{a}^1,...,\tilde{a}^m)=1$.

(ii) *There exist $\tilde{a}^{m+1},...,\tilde{a}^s\in\mathbf{Z}^s$ with $\det(\tilde{a}^1,...,\tilde{a}^s)=1$.*

(iii) *There exist $b_i^j\in\mathbf{Z}$, $i,j=1,...,s$ with*

$$\begin{pmatrix} b_1^1 & b_1^2 & \cdots & b_1^s \\ b_2^1 & b_2^2 & \cdots & b_2^s \\ \vdots & \vdots & \ddots & \vdots \\ b_s^1 & b_s^2 & \cdots & b_s^s \end{pmatrix}(\tilde{a}^1,...,\tilde{a}^s)=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(iv) *There exist $c_i^j\in\mathbf{Z}$, $j=1,...,s$; $i=s+1,...,n$ with*

$$(a^1,...,a^s)=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ c_{s+1}^1 & c_{s+1}^2 & \cdots & c_{s+1}^s \\ \vdots & \vdots & & \vdots \\ c_n^1 & c_n^2 & \cdots & c_n^s \end{pmatrix}(\tilde{a}^1,...,\tilde{a}^s).$$

(v) *There exist $c_i^j\in\mathbf{Z}$, $j=1,...,s$; $i=s+1,...,n$ with*

$$(\overline{a}^1,...,\overline{a}^s)=\begin{pmatrix} c_{s+1}^1 & c_{s+1}^2 & \cdots & c_{s+1}^s \\ \vdots & \vdots & & \vdots \\ c_n^1 & c_n^2 & \cdots & c_n^s \end{pmatrix}(\tilde{a}^1,...,\tilde{a}^s).$$

(vi) *There exist $c_i^j\in\mathbf{Z}$, $j=1,...,s$; $i=s+1,...,n$ with*

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ -c_{s+1}^1 & -c_{s+1}^2 & \cdots & -c_{s+1}^s & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -c_n^1 & -c_n^2 & \cdots & -c_n^s & 0 & \cdots & 1 \end{pmatrix}(a^1,...,a^s)=\begin{pmatrix} a_1^1 & \cdots & a_1^s \\ \vdots & \ddots & \vdots \\ a_1^1 & \cdots & a_1^s \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

(vii) *There exist* $c_i^j \in \mathbf{Z}$, $j = 1, ..., s$; $i = s+1, ..., n$ *with*

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ -c_{s+1}^1 & -c_{s+1}^2 & \cdots & -c_{s+1}^s & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -c_n^1 & -c_n^2 & \cdots & -c_n^s & 0 & \cdots & 1 \end{pmatrix} (a^1, ..., a^m) = \begin{pmatrix} a_1^1 & \cdots & a_1^m \\ \vdots & \ddots & \vdots \\ a_1^1 & \cdots & a_1^m \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

(viii) *There exist* $c_i^j \in \mathbf{Z}$, $j = 1, ..., s$; $i = s+1, ..., n$ *with*

$$\begin{pmatrix} -c_{s+1}^1 & -c_{s+1}^2 & \cdots & -c_{s+1}^s & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -c_n^1 & -c_n^2 & \cdots & -c_n^s & 0 & \cdots & 1 \end{pmatrix} (a^1, ..., a^m) = 0.$$

Proof. (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) is clear. For (iii)$\Longrightarrow$(iv), set

$$\begin{pmatrix} c_{s+1}^1 & \cdots & c_{s+1}^s \\ \vdots & & \vdots \\ c_n^1 & \cdots & c_n^s \end{pmatrix} = \begin{pmatrix} a_{s+1}^1 & \cdots & a_{s+1}^s \\ \vdots & & \vdots \\ a_n^1 & \cdots & a_n^s \end{pmatrix} \begin{pmatrix} b_1^1 & \cdots & b_1^s \\ \vdots & & \vdots \\ b_s^1 & \cdots & b_s^s \end{pmatrix}.$$

(iv)$\Longrightarrow$(v)$\Longrightarrow$(vi)$\Longrightarrow$(vii) $\Longrightarrow$(i) is clear. (vii)$\Longleftrightarrow$(viii) is also clear. □

We now present one typical case that Proposition 5.1 can be applied. Let $I = \{s+1, ..., n\}$. Consider a function f defined by

$$f(x) = g_0(x_1, ..., x_s) + \sum_{k=s+1}^{n} x_k g_k(x_1, ..., x_s) + g(x), \quad g(x) \in (x_{s+1}, ..., x_n)^2.$$

We assume that $S_{I,i} \cap \Gamma_+(f)$ has an integral point for each $i \in I$. Let Δ be a simplicial polyhedrons majorizing $\mathbf{R}_+^n$. Let F be a face of Δ and we let $\gamma = \gamma(F)$. We assume that

$$\text{a point of } F \cap S_{I,i} \cap \mathbf{Z}^n \text{ is a vertex of } F, \quad \forall i \in I.$$

This implies Condition (ii) in Proposition 5.1. We also assume that f_γ can be written in the following form:

$$f_\gamma(x) = G_0(x_1, .., x_s) + \sum_{k=s+1}^{n} x_k G_k(x_1, ..., x_s).$$

Obviously

$$\frac{\partial f_\gamma}{\partial x_i} = \begin{cases} \frac{\partial G_0}{\partial x_i}(x_1, ..., x_s) + \sum_{k=s+1}^n x_k \frac{\partial G_k}{\partial x_i}(x_1, ..., x_s), & \text{for } i = 1, ..., s \\ G_i(x_1, ..., x_s), & \text{for } i = s+1, ..., n \end{cases}$$

So, if

$$\{(x_1, ..., x_s) \mid G_j(x_1, ..., x_s) = 0, \ j = s+1, ..., n\} \subset S(F)$$

then the condition (ii) in Proposition 2.4 is satisfied. So it is possible to construct a polyhedron Δ and a family $\{f_t\}$ such that

- f_t is non-degenerate for $t \neq 0$.
- f_t is Δ-regular.

The simplest examples are Examples 4.3, 4.2.

6. Simultaneous resolution

Definition 6.1. *Let $\mathcal{U}, P$ be a real analytic manifolds and let $p : \mathcal{U} \to P$ be a submersion. We set $U_t = p^{-1}(t)$, for $t \in P$. Let $F : \mathcal{U} \to \mathbf{R}$ be a real analytic function. Let $\Pi : \mathcal{M} \to \mathcal{U}$ be a proper analytic modification. We say that Π gives a simultaneous resolution of F, if the following conditions are satisfied:*

- *$\mathcal{M}$ is nonsingular.*
- *For each point x of $\mathcal{M}$, there is a local coordinate system $y = (y_1, \ldots, y_m)$ centered at x so that*
$$F \circ \Pi(y) = y_1^{\ell_1} \cdots y_m^{\ell_m}$$
and that the restriction of p to $\{y_{j_1} = \cdots = y_{j_p} = 0\}$ is a submersion onto P for each $1 \le j_1 < \cdots < j_p \le m$ with $\ell_{j_1} \cdots \ell_{j_p} \neq 0$.

The proof of Proposition 2.4 implies the following:

Proposition 6.2. *Let $\{f_t\}$ be an analytic family of analytic functions. Let Δ is a nonsingular polyhedron with $\lim_{\varepsilon \to +0} \varepsilon\Delta = \mathbf{R}^n_+$. If $\{f_t\}$ is simultaneously Δ-regular, then $P_\Delta \to \mathbf{R}^n$ gives a simultaneous resolution of $\{f_t\}$.*

It is natural to ask the following question.

Question 6.3. *We set $P = \mathbf{R}^m$. If $\Pi : \mathcal{M} \to \mathcal{U}$ gives a simultaneous resolution of an analytic family $\{f_t : U_t \to \mathbf{R}\}_{t \in \mathbf{R}^m}$ of analytic functions, then we have a family $\{H_t : M_t \to M_0\}_{t \in \mathbf{R}^m}$ of real analytic isomorphisms which trivializes $\{F \circ \pi(x;t)\}_{t \in P}$. Here $F(x;t) = f_t(x)$. After changing $\{H_t\}_{t \in P}$ if necessary, can we expect that $\{H_t\}$ induce a family of homeomorphism germs $\{h_t : U_0 \to U_t\}_{t \in \mathbf{R}^m}$?*

The answer is No!

Example 6.4. *Let $f : (\mathbf{R}^3, 0) \longrightarrow \mathbf{R}$ be a family of function-germs defined by*
$$f_t(x_1, x_2, x_3) = x_1 x_3^3 - x_2^3 + t x_2 x_3^2.$$
Let $\pi : M \longrightarrow \mathbf{R}^3$ be the blow up at (x_2, x_3). It is easy to see $\{f_t\}$ admits a simultaneous resolution by π. In fact, consider a coordinate system (y_1, y_2, y_3) defined by
$$\begin{cases} x_1 &= y_1 \\ x_2 &= y_2 y_3 \\ x_3 &= y_3 \end{cases}$$
and see
$$f_t \circ \pi(y_1, y_2, y_3) = y_3^3(y_1 - y_2^3 + t y_2).$$
Thus f_t admit simultaneous resolution by π. But there are no homeomorphisms between $(\mathbf{R}^3, f_0^{-1}(0), 0)$ and $(\mathbf{R}^3, f_1^{-1}(0), 0)$.

The following example suggests us that the situation would become very complicated in general.

Example 6.5. *Let $f_e : (\mathbf{R}^6, 0) \to (\mathbf{R}, 0)$ be a function defined by*
$$f_e(x) = x_3^2(x_1 x_3^3 - f_{1e}(x_4, x_5, x_6))^2 + (x_2 x_3^4 - f_{2e}(x_4, x_5, x_6))^2$$
where
$$f_{1e}(x_4, x_5, x_6) = (e_1 x_4 - x_5)(e_2 x_4 - x_5)(e_3 x_4 - x_5)$$
$$f_{2e}(x_4, x_5, x_6) = (e_4 x_4 - x_5)(e_5 x_4 - x_5)(e_6 x_4 - x_5) x_6$$
$$e = (e_1, e_2, e_3, e_4, e_5, e_6).$$

Now we consider the blow up $M_1 \to \mathbf{R}^6$ at the ideal (x_3, x_4, x_5, x_6). Consider the coordinate system $(u_1, u_2, u_3, u_4, u_5, u_6)$ defined by

$$\begin{cases} x_i &= u_i \quad i = 1,2,3 \\ x_i &= u_3 u_i \quad i = 4,5,6 \end{cases}$$

Then we obtain $f = u_3^8\left((u_1 - f_{1_e}(v_4, v_5, v_6))^2 + (u_2 - f_{2_e}(v_4, v_5, v_6))^2\right)$. Next we consider the blow up of $M_2 \to M_1$ at the ideal

$$(u_1 - f_{1_e}(u_4, u_5, u_6), u_2 - f_{2_e}(u_4, u_5, u_6), u_3).$$

Consider the coordinate system $(v_1, v_2, v_3, v_4, v_5, v_6)$ defined by

$$\begin{cases} u_i - f_{i_e}(u_4, u_5, u_6) &= v_i v_3 \quad i = 1,2 \\ u_i &= v_i, \qquad i = 3,4,5,6 \end{cases}$$

Then we obtain $f = v_3^{10}(v_1^2 + v_2^2)$, and this family admits a simultaneous resolution. We here remark that

$$\begin{cases} x_i &= v_i v_3 + f_{ie}(v_4, v_5, v_6) \quad i = 1,2 \\ x_3 &= v_3 \\ x_i &= v_3 v_i \qquad i = 4,5,6. \end{cases}$$

If we restrict this to the space defined by $v_3 = 0$, we obtain a map defined by

$$(v_1, v_2, 0, v_4, v_5, v_6) \mapsto (f_{1_e}(v_4, v_5, v_6), f_{2_e}(v_4, v_5, v_6), 0, 0, 0, 0).$$

According to [13]*, this family has infinitely many topologically right-left equivalence classes. So no family of homeomorphisms on M_2 that trivialize the family obtained by pulling-back $\{f_e\}$ does not induce homeomorphisms on $(\mathbf{R}^6, 0)$.*

7. Equisingularity via simultaneous resolution

Let $F : (\mathbf{R}^n \times \mathbf{R}^m, 0) \longrightarrow \mathbf{R}$ be an analytic function and $f_t(x) = F(x;t)$. We assume that f_t admits a simultaneous resolution by the modification $\Pi : \mathcal{M} \to \mathbf{R}^n \times \mathbf{R}^m$. Let $\mathcal{E}$ be the critical locus of Π. We denote by M the strict transform of $(\mathbf{R}^n, 0) \times 0$ and set $E = M \cap \mathcal{E}$. We say that f_t is *equisingular* by Π at $(\mathbf{R}^m, 0)$ if there are homeomorphism-germs $H : (\mathcal{M}, \mathcal{E}) \longrightarrow (M, E) \times (\mathbf{R}^m, 0)$ and $h : (\mathbf{R}^n \times \mathbf{R}^m, 0) \longrightarrow (\mathbf{R}^n \times \mathbf{R}^m, 0)$ with

$$\begin{array}{ccccc} (\mathcal{M}, \mathcal{E}) & \xrightarrow{\Pi} & (\mathbf{R}^n \times \mathbf{R}^m, 0) & \xrightarrow{(F,\mathrm{id})} & \mathbf{R} \times (\mathbf{R}^m, 0) \\ {\scriptstyle H}\downarrow & & {\scriptstyle h}\downarrow & & \| \\ (M, E) \times (\mathbf{R}^m, 0) & \xrightarrow{\pi \times \mathrm{id}} & (\mathbf{R}^n, 0) \times (\mathbf{R}^m, 0) & \xrightarrow{f_0 \times \mathrm{id}} & \mathbf{R} \times (\mathbf{R}^m, 0) \end{array}$$

where $\pi = \Pi|_M$. We can generalize this definition as follows: Let $F : (\mathbf{R}^n \times \mathbf{R}^m, 0) \longrightarrow \mathbf{R}$ be an analytic function and $f_t(x) = F(x;t)$. We consider a composition of modifications

$$\mathcal{M} = \mathcal{M}_k \xrightarrow{\Pi_k} \mathcal{M}_{k-1} \xrightarrow{\Pi_{k-1}} \cdots \longrightarrow \mathcal{M}_2 \xrightarrow{\Pi_2} \mathcal{M}_1 \xrightarrow{\Pi_1} \mathbf{R}^n,$$

and denote the composition by Π. We assume that f_t admits a simultaneous resolution by the modification $\Pi : \mathcal{M} \to \mathbf{R}^n \times \mathbf{R}^m$. Let $\mathcal{E}_i$ be the critical locus of $\Pi_i \circ \cdots \circ \Pi_1$. We denote by M_i the strict transform of $(\mathbf{R}^n, 0) \times 0$ by $\Pi_i \circ \cdots \circ \Pi_1$ and set $\pi_i = \Pi_i|_{M_i}$ and $E_i = M_i \cap \mathcal{E}_i$, for $i = 1, ..., k$. For notational convention, we set $M_0 = \mathbf{R}^n$ and $E_0 = \{0\}$. We say that f_t is *cascade equisingular* by this diagram at $(\mathbf{R}^m, 0)$ if there are homeomorphism-germs

$$H_i : (\mathcal{M}_i, \mathcal{E}_i) \longrightarrow (M_i, E_i) \times (\mathbf{R}^m, 0), \quad i = 0, 1, ..., k$$

satisfying the following commutation diagram:

$$\begin{tikzcd}
(\mathcal{M}_k, \mathcal{E}_k) \arrow[r, "H_k"] \arrow[d, "\Pi_k"'] & (M_k, E_k) \times (\mathbf{R}^m, 0) \arrow[d, "\pi_k \times \mathrm{id}"] \\
(\mathcal{M}_{k-1}, \mathcal{E}_{k-1}) \arrow[r, "H_{k-1}"] \arrow[d, "\Pi_{k-1}"'] & (M_{k-1}, E_{k-1}) \times (\mathbf{R}^m, 0) \arrow[d, "\pi_{k-1} \times \mathrm{id}"] \\
\vdots \arrow[d, "\Pi_2"'] & \vdots \arrow[d, "\pi_2 \times \mathrm{id}"] \\
(\mathcal{M}_1, \mathcal{E}_1) \arrow[r, "H_1"] \arrow[d, "\Pi_1"'] & (M_1, E_1) \times (\mathbf{R}^m, 0) \arrow[d, "\pi_1 \times \mathrm{id}"] \\
(\mathbf{R}^n \times \mathbf{R}^m, 0) \arrow[r, "H_0"] \arrow[d, "F \times \mathrm{id}"'] & (\mathbf{R}^n, 0) \times (\mathbf{R}^m, 0) \arrow[d, "f_0 \times \mathrm{id}"] \\
\mathbf{R} \times (\mathbf{R}^m, 0) \arrow[r, equal] & \mathbf{R} \times (\mathbf{R}^m, 0)
\end{tikzcd}$$

Proposition 7.1. *Let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, $t \in (\mathbf{R}^m, 0)$, be a family of analytic functions. Let U be a neighborhood of $(0,0)$. Suppose that a modification $\Pi : \mathcal{M} \to U$ gives a simultaneous resolution of $\{f_t\}$. Let $\mathcal{E}$ denote the critical locus of Π. We assume that $\mathcal{E}$ is normal crossing divisor. We denote by X the strict transform of $(\mathbf{R}^n, 0) \times 0$ and by π the restriction of Π to X. We set $E = \mathcal{E} \cap X$. If there exist a homeomorphism-germ $H : (\mathcal{E}, E) \to E \times (\mathbf{R}^m, 0)$ and a homeomorphism-germ $h : \Pi(\mathcal{E}) \to \pi(E) \times (\mathbf{R}^m, 0)$ such that $h \circ \Pi = (\pi \times \mathrm{id}) \circ H$, then $\{f_t\}$ is equisingular by Π.*

So if we have a situation in which we can apply Thom's second isotopy lemma (Theorem in (8.6) of [14], or II Theorem (5.8) of [20]) for the following diagram, we have a criterion for equisingularity.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "H"] \arrow[d, "\Pi"'] & E \times (\mathbf{R}^m, 0) \arrow[d, "\pi \times \mathrm{id}"] & \\
\Pi(\mathcal{E}) \arrow[r, "h"] & \pi(E) \times (\mathbf{R}^m, 0) \arrow[r] & (\mathbf{R}^m, 0)
\end{tikzcd}$$

Corollary 7.2. *Assume that $\Pi(\mathcal{E}) = 0 \times (\mathbf{R}^m, 0)$ (i.e., $\pi(E) = 0$). If Π gives a simultaneous resolution of $\{f_t\}$, then $\{f_t\}$ is equisingular.*

The proof of Proposition 7.1 is based on the following.

Lemma 7.3. *Let $F : M_1 \to M_2$ be a homeomorphism between metric spaces, and $g_i : M_i \to N_i$, $i = 1, 2$, proper maps between metric spaces. If there is a map $f : N_1 \to N_2$ with $f \circ g_1 = g_2 \circ F$, then f is continuous.*

Proof. Let U be an open subset of N_2. If $f^{-1}(U)$ is not open, then there are a point $P \in f^{-1}(U)$ and a sequence $\{P_n\}$ in $N_1 - f^{-1}(U)$ tends to P. We remark that $f(P_n) \notin U$ for any n. Consider a sequence $\{Q_n\}$ in M_1 with $g_1(Q_n) = P_n$. Since g_1 is proper, there is a convergent subsequence of $\{Q_n\}$. We may write this subsequence by $\{Q_n\}$, by economy of notation. The image of this by $g_2 \circ F$ is convergent to $f(P)$. Since U is open, there is a number N so that $f(P_n) \in U$ for $n \geq N$, and this is a contradiction. $\square$

Now we consider the tower of the polyhedrons: $\{\Delta_k \to \Delta_{k-1} \to \cdots \to \Delta_1 \to \mathbf{R}^n_{\geq}\}$. We consider cascade equisingularity for the diagram:

$$P_{\Delta_k} \times (\mathbf{R}^m, 0) \to P_{\Delta_{k-1}} \times (\mathbf{R}^m, 0) \to \cdots \to P_{\Delta_1} \times (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0) \times (\mathbf{R}^m, 0).$$

Because of Lemma 7.3, it is enough to analyse everything after restricting along the exceptional sets. After this we need to describe the critical locus of the restriction of Π_i to $Z_{F_i}(f)$ where F_i is a face of Δ_i. The following lemmas are a consequence of Lemma 1.2, 1.3 and the implicit function theorem.

Lemma 7.4. *The critical point set of the restriction of* $P_{F_i} \to P_{F_{i-1}}$ *to* $T_{F_i} \cap Z_{\Delta_i}(f)$ *is defined as the zero of* $\delta_a f_\gamma$, $a \in F_{i-1}^{\perp}$.

Lemma 7.5. *The critical point set of the restriction of* $P_{F_i} \to P_{F_{i-1}}$ *to* $E_{F_i} \cap Z_{\Delta_i}(f)$ *is defined as the zero of* $\delta_a f_\gamma$, $a \in F_{i-1}^{\perp}$ *with* $a \notin \{e_1, ..., e_n\}$, *and* $\frac{\partial f_\gamma}{\partial x_j}$ *for* j *with* $e_j \in F_{i-1}^{\perp}$.

Now we concentrate the case $n = 3$, $m = 1$, that is, a 1-parameter family of functions $f_t(x, y, z) = f(x, y, z; t)$ in three variables (x, y, z). If we use all together below, we obtain a sufficient condition for cascade equisingularity. We assume that the Newton polyhedron of f_t is constant, f_t are simultaneously non-degenerate, and that $P_{\Delta_k} \to \mathbf{R}^3$ gives a simultaneous resolution of f_t. Let E_i be the critical locus of $P_{\Delta_i} \to \mathbf{R}^n$. We seek a condition which implies cascade equisingularity. To do this it is enough to seek a condition which assure the existence of Thom regular stratifications for map $T_{F_i} \to T_{F_{i-1}}$. We are going to construct these stratifications of $E_i \times \mathbf{R}$ which are refinements of the stratification of $E_i \times \mathbf{R}$ induced by toric stratifications of P_Δ.

Let F_i be a face of Δ_i and let F_{i-1} be a face of Δ_{i-1}. We assume that the map $\Delta_i \to \Delta_{i-1}$ send F_i to F_{i-1}.

If $\dim F_i = \dim F_{i-1}$, then the natural map of T_{F_i} to $T_{F_{i-1}}$ is an isomorphism. So a homeomorphism of $T_{F_i} \times \mathbf{R}$ induces a homeomorphism of $T_{F_{i-1}} \times \mathbf{R}$.

We next consider the case $\dim F_{i-1} = 0$. In this case, we have $P_{F_{i-1}} \times \mathbf{R} \simeq \mathbf{R}$. Then we consider a stratification of T_{F_i} whose strata maps onto $T_{F_i-1} \times \mathbf{R} \simeq \mathbf{R}$ submersively. Thus a homeomorphism of $T_{F_i} \times \mathbf{R}$ obtained by integrating a vector field constructed in the proof of the isotopy lemma induces a homeomorphism of $T_{F_{i-1}} \times \mathbf{R}$. So it is enough to construct a regular stratification of T_{F_i}. This is always possible when $f_t(x, y, z)$ is simultaneously non-degenerate.

For the remaining case, that is, the case $\dim F_i = 2$ and $\dim F_{i-1} = 1$, we have the following.

Proposition 7.6. *We assume that* $\dim F_i = 2$ *and* $\dim F_{i-1} = 1$. *Let* a *denote a vector with* $a \in {F_{i-1}}^{\perp}$. *If there is a stratification of* $P_{F_i} \times \mathbf{R}$ *with the following properties, then there is a stratification of* $P_{F_{i-1}}$ *such that* $T_{F_i} \times \mathbf{R} \to T_{F_{i-1}} \times \mathbf{R}$ *is a Thom map.*

(i) *For a* 2*-dimensional stratum* X *in* $T_{F_i} \times \mathbf{R}$, $f_{t\gamma}|_X = 0$ *and* $\delta_a f_{t\gamma}$ *does not vanish on* X.

(ii) *For a* 1*-dimensional stratum* X *in* $T_{F_i} \times \mathbf{R}$, $f_{t\gamma}|_X = \delta_a f_{t\gamma}|_X = 0$ *and* $\delta_a \delta_a f_{t\gamma}$ *does not vanish on* X.

Proof. We can consider the restriction of δ_a to T_{F_i} as a vector field tangent to fibers of $T_{F_i} \to T_{F_{i-1}}$. We use the implicit function theorem. For a 2-dimensional stratum X, the condition (i) implies the restriction of the map $T_{F_i} \times \mathbf{R} \to T_{F_{i-1}} \times \mathbf{R}$ to X is a submersion, since the strict transform of f intersects T_{F_i} transversely. For a 1-dimensional stratum X, the condition (ii) implies the image of X by the map $T_{F_i} \times \mathbf{R} \to T_{F_{i-1}} \times \mathbf{R}$ is a manifold. So we can define a stratification of $T_{F_{i-1}} \times \mathbf{R}$ so that the map $T_{F_i} \times \mathbf{R} \to T_{F_{i-1}} \times \mathbf{R}$ is stratified. The remaining assertions (regularities, etc.) are not difficult to see. □

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